

# Maximal volume behind horizons without curvature singularity

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(Dated: February 20, 2017)

The black hole information paradox is related to the area of event horizon, and potentially to the volume and singularity behind it. One example is the complexity/volume duality conjectured by Stanford and Susskind. Accepting the proposal of Christodoulou and Rovelli, we calculate the maximal volume inside regular black holes, which are free of curvature singularity, in asymptotically flat and anti-de Sitter spacetimes respectively. The complexity/volume duality is then applied to anti-de Sitter regular black holes. We also present an analytical expression for the maximal volume outside the de Sitter horizon.

## I. INTRODUCTION

Information is obscured behind the horizon as reflected in the black hole information paradox [1]. There have been several solutions to this paradox, and there will be more until the fulfilment of the quantum theory of gravity. In most solutions to the paradox, one should take into consideration of the area, volume and singularity of black holes. For example, in reference [2], Stanford and Susskind proposed that the complexity of a state on the anti-de Sitter (AdS) boundary is proportional to the maximal spatial volume of the Einstein-Rosen bridge anchored at the boundary state. This refines Susskind's earlier conjecture [3] that the complexity is proportional to the length of the Einstein-Rosen bridge. And it is later superseded by an elaborated conjecture [4, 5] relating the complexity to the action of a Wheeler-DeWitt patch. Among the three conjectures, we feel that the complexity/volume duality conjecture is visually the most elegant though not perfect yet.

Apparently independent of the above stream, Christodoulou and Rovelli define the volume of a black hole as the volume of maximal spacelike hypersurface bounded by the event horizon [6]. This was initially illustrated in details with spherically symmetric spacetime, such as the Schwarzschild and the Reissner-Nordström (RN) black holes. Later, more efforts were made on the extension to the Kerr black hole [7], the relations between volume and entropy [8–10], and the effects of Hawking radiation [11–13].

So far the main attention has been paid on interior volume of black holes with curvature singularities, except for a very recent work [14] which turned to non-commutative black holes. As emphasized by references [15, 16], to resolve the information paradox, despite the potentially important role of volumes, we should not ignore the singularity. In the present paper, we will explore the case without curvature singularity, specifically two examples: the volume inside regular black holes in sections II and III, and the volume outside the de Sitter (dS) horizon in section IV.

Although developed independently, the black hole volume defined by Christodoulou and Rovelli ought to be useful for the complexity/volume duality in the infinite-time limit [2], as we will emphasize in section III. Remember that the complexity is defined for a state on the AdS boundary. Therefore, to apply the complexity/volume duality, we should restrict our investigations to black holes in asymptotically AdS spacetime. But in section II the examples collected from literature [17–22] are asymptotically flat at spatial infinity. Fortunately, an asymptotically AdS regular black hole was constructed recently by Fan in reference [23], see more solutions in reference [24]. Partially inspired by his work, in appendix A, we will demonstrate a prescription to switch on cosmological constant in a class of solutions for the Einstein gravity. This prescription can be applied to all solutions in section II to get their AdS counterparts. In section III, to such asymptotically AdS solutions we will apply Christodoulou and Rovelli's definition of black hole volume together with Stanford and Susskind's conjecture of complexity/volume duality. Some loose ends will be discussed in section V.

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## II. VOLUME INSIDE REGULAR BLACK HOLES

Throughout this paper, we will work in units in which  $G_N = \hbar = c = 1$ , and restrict our study to  $(3 + 1)$ -dimensional spherical static spacetimes described by the line element

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2 \quad (1)$$

where  $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ . Using the advanced time  $v = t + \int f^{-1}(r)dr$ , it can be rewritten as

$$ds^2 = -f(r)dv^2 + 2dvdr + r^2 d\Omega^2. \quad (2)$$

A specific form of the lapse function  $f(r)$  dictates a special spacetime. For instance, the RN spacetime is determined by a lapse function of the form

$$f(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2}. \quad (3)$$

In the chargeless case  $q = 0$ , it reduces to the Schwarzschild spacetime.

It is well known that both the Schwarzschild and the RN black holes have a curvature singularity at the spatial origin  $r = 0$  inside the event horizon. In contrast, regular black holes retain event horizons but get rid of curvature singularities. In reference [22], Ayon-Beato and Garcia showed in a specific example that regular black holes are solutions to Einstein equations coupled to nonlinear electrodynamics. Since then many solutions of regular black hole have been constructed in the literature.

Lapse function	Extremality condition	Reference	Tag
$f(r) = 1 - \frac{2mr^2}{(r^2+q^2)^{3/2}}$	$\frac{q}{m} \approx 0.77$	[17, 18]	RBHa
$f(r) = 1 - \frac{2mr^2}{r^3+q^3}$	$\frac{q}{m} \approx 1.06$	[19]	RBHb
$f(r) = 1 - \frac{2m}{r} \left(1 - \tanh \frac{q^2}{2mr}\right)$	$\frac{q}{m} \approx 1.05$	[20]	RBHc
$f(r) = 1 - \frac{4m}{\pi r} \left(\arctan \frac{8mr}{\pi q^2} - \frac{8m\pi q^2 r}{64m^2 r^2 + \pi^2 q^4}\right)$	$\frac{q}{m} \approx 1.07$	[21]	RBHd
$f(r) = 1 - \frac{2mr^2}{(r^2+q^2)^{3/2}} + \frac{q^2 r^2}{(r^2+q^2)^2}$	$\frac{q}{m} \approx 0.63$	[22]	RBHe

Table I. The lapse function of some regular black holes asymptotic to flat spacetime at  $r \rightarrow \infty$ . To facilitate comparison, we have transformed the model parameters into the mass  $m$  and the charge  $q$ . Please refer to original references for details.

For several regular black holes [17–22], the lapse functions are summarized in table I. At the spatial infinity  $r \rightarrow \infty$ , all of them behave as  $f(r) \rightarrow 1$ , corresponding to asymptotically flat spacetime. There are two parameters: the mass  $m$  and the charge  $q$ . As long as  $q \neq 0$ , these black holes are free of singularity. When the charge-to-mass ratio is smaller than the extremal value in table I, each of them has two horizons  $r = r_{\pm}$ , which shrink into a single one under the extremality condition. In this section, we will compute the maximal volume of these black holes by following Christodoulou and Rovelli's recipe. As they demonstrated in reference [6], the maximal volume bounded by horizons of a spherical black hole can be evaluated via the integral

$$V_{\max} = 4\pi \int_{r_-}^{r_+} dr \frac{r^4}{\sqrt{A_{\min}^2 + r^4 f(r)}} + 4\pi \int_0^{r_-} dr \frac{r^2}{\sqrt{f(r)}}. \quad (4)$$

Here  $A_{\min}$  is the maximum of  $\sqrt{-r^4 f(r)}$  in the range  $r_- < r < r_+$ . In other words,  $A_{\min}^2$  is the minimal constant that guarantees the expression nonnegative under the radical sign.

In the expression (4), the first term is often divergent. The second term, which will be denoted by  $V_{\text{in}}$ , yields often a negligible contribution. One can interpret the second term as the volume bounded by the inner horizon, and the first term naively as the volume between inner and outer horizons. In most cases, the interval  $(r_-, r_+)$  is finite, then the volume at large  $v$ , as was proven in reference [6], grows linearly with advanced time,

$$V_{\max} \sim 4\pi A_{\min} v. \quad (5)$$

It is interesting to note that this relation implies the consistence between two conjectures in references [2, 3], which state that the complexity is proportional to the length and the maximal volume of the Einstein-Rosen bridge respectively.

Especially, for the Schwarzschild black hole [6], one finds  $V_{\max} \sim 3\sqrt{3}\pi m^2 v$  at large  $v$ . For more complicated solutions, usually one cannot calculate  $A_{\min}$  analytically, but it is feasible to numerically evaluate the function  $\sqrt{-r^4 f(r)}$  and seek for its maximum

$$\frac{V_{\max}}{4\pi v} \sim \max \left[ \sqrt{-r^4 f(r)} \right]. \quad (6)$$

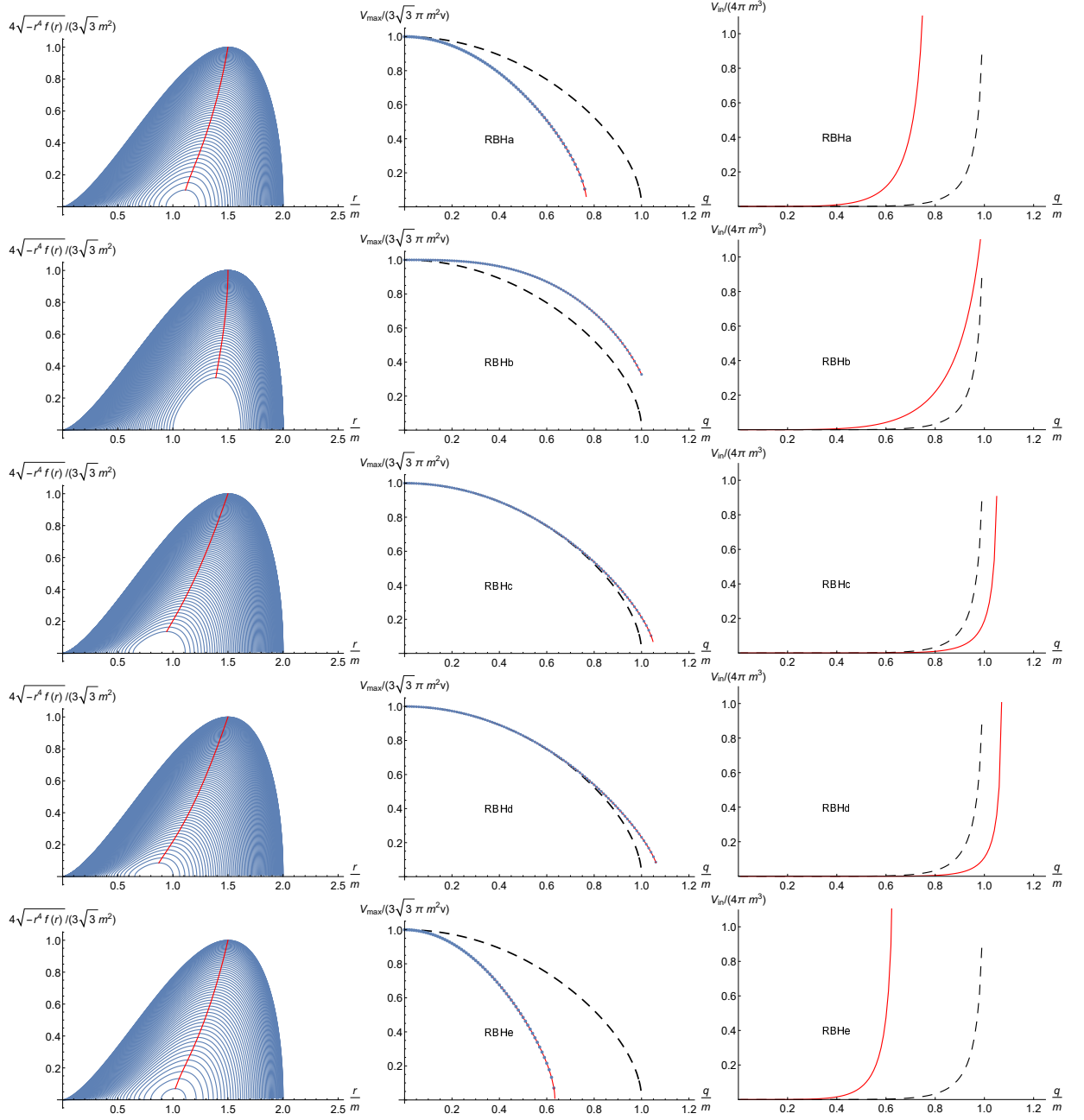


Figure 1. (color online). Left and middle panels: the growth rate of maximal volume at large  $v$ , normalized by the Schwarzschild limit  $3\sqrt{3}\pi m^2$ . Right panels: the volume inside the inner horizon, normalized by  $4\pi m^3$ . Red and blue curves from top to bottom, tagged as RBHa, RBHb, RBHc, RBHd, RBHe accordingly, correspond to the black hole solutions in table I row by row. Black dashed lines depict results of the RN black hole for comparison.

Using the parameter  $m$  to non-dimensionalize all quantities and varying the other parameter  $q$  from zero to the extremal value, we have performed the numerical evaluation for all of the black holes in table I. The results are shown in figure 1.

To facilitate comparison, we tagged the five regular black hole solutions with RBHa, RBHb, etc in table I and figure 1 correspondingly. In every row of figure 1, each value of  $q$  gives a blue curve in the left panel and a blue point in the middle panel. As the charge  $q$  varies from zero to its extremal value continuously, the maximum (6) decreases along the red trajectory in both left and middle panels. Therefore, according to Christodoulou and Rovelli's definition of black hole volume, we can say that, given the mass  $m$  of a black hole, the less charge it has, the faster it grows in volume at large  $v$ . This behavior is similar to that of RN black holes, as depicted by black dashed lines in middle panels of figure 1.

We end this section by some comments on the volume  $V_{\text{in}}$  bounded by the inner horizon. In the extremal case  $r_+ = r_-$ , the

first term of formula (4) vanishes and the second term becomes divergent. Then the leading-order contribution is

$$\begin{aligned}
V_{\text{in}} &= 4\pi \int_0^{r_-} dr \frac{r^2}{\sqrt{f(r)}} \\
&\sim \lim_{r_+ \rightarrow r_-} 4\pi \int_0^{r_-} dr \frac{r^2}{\sqrt{\frac{1}{2}f''(r_-)(r_+ - r)(r_- - r)}} \\
&\sim \lim_{r_+ \rightarrow r_-} -\frac{4\sqrt{2}\pi r_-^2}{\sqrt{f''(r_-)}} \ln(r_+ - r_-).
\end{aligned} \tag{7}$$

Hereafter primes denote derivatives with respect to the indicated variables. For non-extremal values of  $q$ , we have evaluated  $V_{\text{in}}$  numerically in right panels of figure 1. In all examples, the curve of  $V_{\text{in}}$  becomes very steep as  $q$  increases in the near-extremal region.

### III. ADS REGULAR BLACK HOLES AND COMPLEXITY/VOLUME DUALITY

This section is motivated by the observation that relation (6) coincides with equation (2.5) in reference [2] where  $V$  is the volume of the infinite-time Einstein-Rosen bridge. The coincidence occurs when we restrict Christodoulou and Rovelli's proposal to asymptotically AdS black holes and take the infinite-time limit of Stanford and Susskind's conjecture. We intend to put the two ingredients together in regular black holes, but there are two obstacles.

First, in the literature, most solutions of regular black holes asymptotic to a flat spacetime at  $r \rightarrow \infty$  like the examples in section II. This obstacle is overcome in appendix A, where we demonstrate that in the Einstein gravity, from an asymptotically flat solution of the spherical static form (1), one can get an asymptotically AdS solution by replacing the lapse function  $f(r)$  with  $f(r) + r^2/\ell^2$ , leaving intact the electromagnetic field. The AdS radius  $\ell$  is related to the cosmological constant via  $\Lambda = -3/\ell^2$ . Applying this prescription to solutions in table I, we wrote down their AdS counterparts in table II, tagged as AdS-RBH<sub>a</sub>, AdS-RBH<sub>b</sub>, etc. Note that AdS-RBH<sub>a</sub> and AdS-RBH<sub>b</sub> were firstly constructed in references [23, 24].

Lapse function	Tag
$f(r) = \frac{r^2}{\ell^2} + 1 - \frac{2mr^2}{(r^2+q^2)^{3/2}}$	AdS-RBH <sub>a</sub>
$f(r) = \frac{r^2}{\ell^2} + 1 - \frac{2mr^2}{r^3+q^3}$	AdS-RBH <sub>b</sub>
$f(r) = \frac{r^2}{\ell^2} + 1 - \frac{2m}{r} \left(1 - \tanh \frac{q^2}{2mr}\right)$	AdS-RBH <sub>c</sub>
$f(r) = \frac{r^2}{\ell^2} + 1 - \frac{4m}{\pi r} \left(\arctan \frac{8mr}{\pi q^2} - \frac{8m\pi q^2 r}{64m^2 r^2 + \pi^2 q^4}\right)$	AdS-RBH <sub>d</sub>
$f(r) = \frac{r^2}{\ell^2} + 1 - \frac{2mr^2}{(r^2+q^2)^{3/2}} + \frac{q^2 r^2}{(r^2+q^2)^2}$	AdS-RBH <sub>e</sub>

Table II. The lapse function of some regular black holes asymptotic to the AdS spacetime at  $r \rightarrow \infty$ . They correspond line by line to the asymptotically flat solutions in table I.

Second, in the complexity/volume duality, there is a length scale undetermined. As mentioned in reference [5], the original complexity/volume duality [2] stated that the complexity of the boundary state is proportional to the spatial volume  $V_{\text{max}}$  of a maximal slice behind the horizon,

$$\mathcal{C} \sim \frac{V_{\text{max}}}{\ell_c}, \tag{8}$$

where  $\ell_c$  is a length scale that has to be chosen appropriately for the configuration. For Schwarzschild-AdS black holes,  $\ell_c$  is typically chosen as the AdS radius for large black holes and the Schwarzschild radius for small black holes [5]. However, there is no universal form of  $\ell_c$ , especially for intermediate-sized black holes and charged black holes that we want to study. In fact, that was why Brown et. al. turned to the complexity/action duality in references [4, 5].

In this paper, we will tentatively circumvent the second obstacle by exploring the possibility that

$$\ell_c = \sqrt{\frac{\alpha r_+}{T}}, \tag{9}$$

where  $\alpha$  is a constant to be determined soon,  $r_+$  is the radius of outer horizon, and the Hawking temperature  $T = f'(r_+)/(4\pi)$ . For Schwarzschild-AdS black holes, one may check that, as has been expected,  $\ell_c \sim 2\ell\sqrt{\alpha\pi}/3$  if  $r_+ \gg \ell$  and  $\ell_c \sim 2r_+\sqrt{\alpha\pi}$  if

$r_+ \ll \ell$ . We will fix  $\alpha$  by demanding that small Schwarzschild-AdS black holes saturate the complexity growth bound proposed in references [4, 5]

$$\frac{d\mathcal{C}}{dv} \leq \frac{2}{\pi} \int T dS, \quad (10)$$

and then check this bound and ansatz (9) with asymptotically AdS regular black holes. This is far from an exhaustive investigation, which would involve the finite-time Einstein-Rosen bridge and other geometries beyond the scope of this paper.

For simplicity, we replace the integral in inequality (10) with  $2TS$ . They are approximately the same for small black holes, but not for near-extremal large black holes [5]. Therefore, our results for large black holes in the near-extremal regime will have some uncertainties and should be taken with caution. Recalling that the Bekenstein-Hawking entropy  $S = \pi r_+^2$ , one can rewrite inequality (10) concretely as

$$\frac{V_{\max}}{v} \lesssim 2r_+^2 \sqrt{\frac{\alpha r_+ f'(r_+)}{\pi}}. \quad (11)$$

Limited to small Schwarzschild-AdS black holes, one has  $r_+ \sim 2m$ ,  $r_+ f'(r_+) \sim 1$  and  $V_{\max} \sim 3\sqrt{3}\pi m^2 v$  at large  $v$ . In this limit, the complexity growth bound can be well saturated if and only if  $\alpha = (3\pi/4)^3$ .

Normalized to the Schwarzschild limit, the complexity growth bound now takes the form of a bound on the volume of black holes,

$$\frac{V_{\max}}{3\sqrt{3}\pi m^2 v} \lesssim \frac{r_+^2}{4m^2} \sqrt{r_+ f'(r_+)}. \quad (12)$$

This inequality can be numerically tested with black holes in table II. The results are illustrated in figure 2, where the left hand side of inequality (12) is depicted by red solid curves, and the right hand side by black dashed curves. For comparison, in the bottom panels we presented results for the RN-AdS black hole, which is characterized by the lapse function

$$f(r) = \frac{r^2}{\ell^2} + 1 - \frac{2m}{r} + \frac{q^2}{r^2}. \quad (13)$$

In all of the left panels,  $\ell = 2m$ , which is the case of intermediate-sized black holes. In right panels,  $\ell = 100m$ , the case of large black holes. In every panel, we can see the red solid curve follows the black dashed curve closely. This suggests that the complexity growth bound is roughly saturated in all cases. Although not to be taken too seriously, it is quite impressive that the red solid curve lies slightly below the black dashed curve in most panels, which means the bound is not violated. The only exception is AdS-RBHe with  $\ell/m = 100$ , which appears to violate the bound. In section V we will give some important comments on these results.

#### IV. VOLUME OUTSIDE DS HORIZON

Another distinct example of singularity-free spacetime with an event horizon is the dS spacetime, which is given by the lapse function

$$f(r) = 1 - \frac{r^2}{\ell^2}. \quad (14)$$

In this section, the dS radius  $\ell$  is related to the cosmological constant via  $\Lambda = 3/\ell^2$ .

As a warm-up, let us first consider the volume inside the dS horizon, which is conventionally calculated by integrate the volume measure of metric (1) at constant time  $dt = 0$ , that is

$$V_{\text{in}} = 4\pi \int_0^\ell dr \frac{r^2}{\sqrt{f(r)}} = \pi^2 \ell^3. \quad (15)$$

Agreeably, the same result can be also obtained with Christodoulou and Rovelli's recipe by maximizing the integral

$$4\pi \int_0^\ell dr \frac{r^4}{\sqrt{A^2 + r^4 f(r)}} \quad (16)$$

in the range  $0 < r < \ell$ .

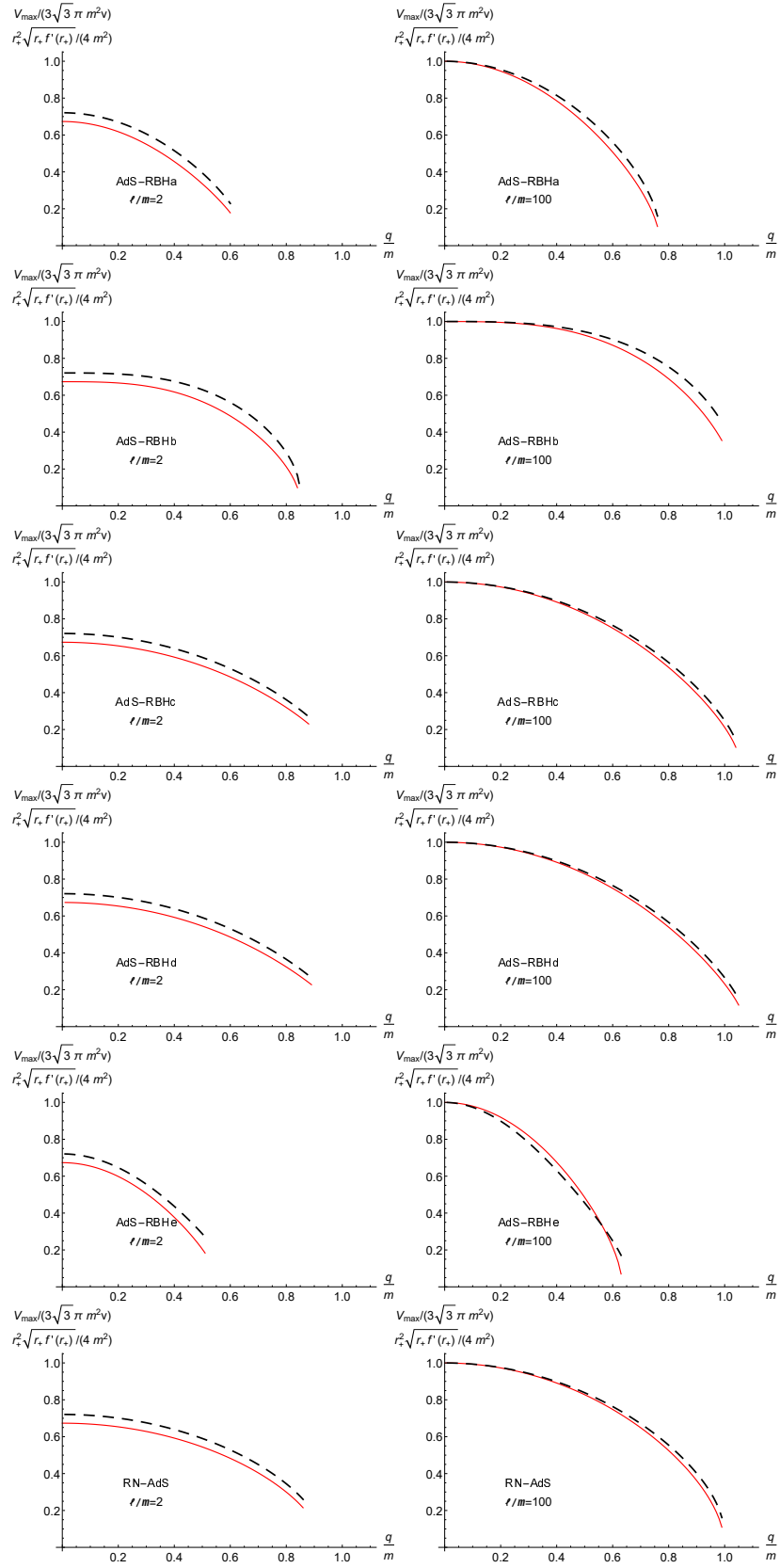


Figure 2. (color online). Red solid lines depict the left hand side of inequality (12), the growth rate of maximal volume at large  $v$  normalized by the Schwarzschild limit  $3\sqrt{3}\pi m^2$ . Black dashed lines display the right hand side of inequality (12), the normalized bound on the volume of black holes. From top to bottom, tagged as AdS-RBH a, AdS-RBH b, AdS-RBH c, AdS-RBH d, AdS-RBH e, RN-AdS accordingly, corresponds to the black hole solutions in table II and solution (13) row by row. The AdS radius  $\ell$  is set to  $2m$  in all left panels, and  $100m$  in all of the right panels.

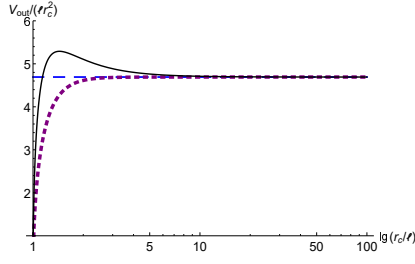


Figure 3. (color online). The volume (19) outside dS horizon in terms of the cutoff radius  $r_c$ . The black solid line, the purple dotted line and the blue dashed line depict numerical result of the first, the third and the last lines of equation (19) respectively.

Applying Christodoulou and Rovelli's method to the outside of dS horizon, one would find a volume formally

$$V_{\text{out}} = \max \left[ 4\pi \int_{\ell}^{\infty} dr \frac{r^4}{\sqrt{A^2 + r^4 f(r)}} \right]. \quad (17)$$

The second step is to determine  $A$  by maximizing  $\sqrt{-r^4 f(r)}$  in the range  $r > \ell$ . Unfortunately, it is divergent in the limit  $r \rightarrow \infty$ . Even if one sets  $A$  to a finite value by hand, the integrand is divergent linearly as  $r \rightarrow \infty$ . The estimation (5) is also ruined here, because deriving (5) relies on the condition that  $r$  spans a finite region.

To proceed, we revise this volume to a tamable form

$$\begin{aligned} V_{\text{out}} &= \lim_{r_c \rightarrow \infty} \max \left[ 4\pi \int_{\ell}^{r_c} dr \frac{r^4}{\sqrt{A^2 + r^4 f(r)}} \right] \\ &= \lim_{r_c \rightarrow \infty} 4\pi \int_{\ell}^{r_c} dr \frac{r^4}{\sqrt{r^4 f(r) - r_c^4 f(r_c)}}. \end{aligned} \quad (18)$$

Here we have introduced a cutoff radius  $r_c$  which is pushed to infinity after maximization. At large  $r_c$ , the volume outside dS horizon grows quadratically with cutoff radius,

$$\begin{aligned} V_{\text{out}} &\sim 4\pi \int_{\ell}^{r_c} dr \frac{r^4}{\sqrt{r^4 f(r) - r_c^4 f(r_c)}} \\ &\sim 4\pi \int_{\ell}^{r_c} dr \frac{\ell r^4}{\sqrt{r_c^6 - r^6}} \\ &= \frac{2\pi^{3/2} \ell r_c^2 \Gamma(\frac{5}{6})}{\Gamma(\frac{1}{3})} - \frac{4\pi \ell^6}{5r_c^3} {}_2F_1\left(\frac{1}{2}, \frac{5}{6}; \frac{11}{6}; \frac{\ell^6}{r_c^6}\right) \\ &\sim \frac{2\pi^{3/2} \ell r_c^2 \Gamma(\frac{5}{6})}{\Gamma(\frac{1}{3})}. \end{aligned} \quad (19)$$

This equation involves the gamma function and the hypergeometric function. In figure 3 we have numerically evaluated the first, the third and the last lines of this equation, depicted by black solid, purple dotted and blue dashed lines respectively. They coincide at large  $r_c$ .

In the second line of equation (19), we kept only the quadratic term in the lapse function. Therefore, we expect this equation holds not only for pure dS spacetime, but also for asymptotically dS spacetime with black holes inside the dS horizon.

## V. DISCUSSION

As proposed by Christodoulou and Rovelli, the volume of a black hole can be defined as the volume of maximal spacelike hypersurface bounded by the event horizon [6]. In the present paper, we have investigated the implications of this proposal for some spacetimes with event horizons but free of singularity: asymptotically flat regular black holes, asymptotically AdS regular black holes, and the dS spacetime. The asymptotically AdS regular black holes are generated by a prescription demonstrated in appendix A, which should be useful for studying solutions to Einstein equations coupled to nonlinear electrodynamics.

Restricted to regular black holes asymptotic to the AdS spacetime at spatial infinity, in section III we explored the conjectures of complexity/volume duality [2] and complexity growth bound [4, 5], assuming a length scale (9). In figure 2, we found the



bound is nearly saturated in all examples and slightly violated in only one case. These behaviors are encouraging. In spite of the good behaviors, we should warn the readers that the length scale (9) is ad hoc, and our primitive exploration in section III relies on the estimation  $\int T dS \sim 2TS$ . Both of these approximations should be refined in an exhaustive study with RN-AdS and Kerr-AdS black holes as well as other geometries, which is out of the scope this paper. We plan to return to this question in a future work.

### ACKNOWLEDGMENTS

This work is supported by the National Natural Science Foundation of China (Grant No. 91536218), and in part by the Science and Technology Commission of Shanghai Municipality (Grant No. 11DZ2260700). The authors would like to thank Yu-Chen Ding for helpful discussions.

### Appendix A: A proper gesture to switch on cosmological constant

Many solutions of regular black hole have been constructed in the literature, especially after Ayon-Beato and Garcia proposed the nonlinear-electrodynamics interpretation in reference [22]. Most of them are of the spherical static form (1) on which we focus in this paper. As we will demonstrate in this appendix, in the Einstein gravity, there is a simple prescription to switch on cosmological constant in this class of solutions, including both regular and singular ones.

For this class of solutions, the symmetry implies that the nonvanishing components of electromagnetic strength are  $F_{tr} = -F_{rt}$  and  $F_{\theta\phi} = -F_{\phi\theta}$ . For clarity, let us introduce notations  $E \equiv F_{tr}$  and  $B \equiv F_{\theta\phi}$ . Furthermore, raising the indices with metric (1), we find  $F^{tr} = -F^{rt} = -E$  and  $F^{\theta\phi} = -F^{\phi\theta} = B/(r^4 \sin^2 \theta)$ . The key point is that the lapse function  $f(r)$  does not appear in the expression of  $F_{\mu\nu}$  and  $F^{\mu\nu}$ .

In the Einstein gravity, when the matter field is given by nonlinear electrodynamics, from the full action [20, 24]

$$I = \frac{1}{16\pi} \int d^4x \sqrt{-g} [R - \mathcal{L}(\mathcal{F})] \quad (\text{A1})$$

in which  $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$ , one can write down the Einstein equations

$$G_\mu^\nu = 2 \left( \mathcal{L}_\mathcal{F} F_{\mu\lambda} F^{\nu\lambda} - \frac{1}{4} \delta_\mu^\nu \mathcal{L} \right) \quad (\text{A2})$$

as well as the nonlinear electrodynamic equations

$$\partial_\mu (\mathcal{L}_\mathcal{F} F^{\mu\nu}) + \Gamma_{\mu\lambda}^\mu \mathcal{L}_\mathcal{F} F^{\lambda\nu} = 0. \quad (\text{A3})$$

Substituting line element (1) and the expression of electromagnetic strength, we can put them in the form

$$\begin{aligned} \frac{f''}{2} + \frac{f'}{r} &= 2 \left( -E^2 \mathcal{L}_\mathcal{F} - \frac{1}{4} \mathcal{L} \right), \\ \frac{f'}{r} + \frac{f-1}{r^2} &= 2 \left( \frac{B^2}{r^4 \sin^2 \theta} \mathcal{L}_\mathcal{F} - \frac{1}{4} \mathcal{L} \right), \end{aligned} \quad (\text{A4})$$

$$\begin{aligned} \partial_r (E \mathcal{L}_\mathcal{F}) + \frac{2}{r} E \mathcal{L}_\mathcal{F} &= 0, \\ \partial_t (E \mathcal{L}_\mathcal{F}) &= 0, \\ \partial_\phi (B \mathcal{L}_\mathcal{F}) &= 0, \\ \partial_\theta \left( \frac{B}{\sin^2 \theta} \mathcal{L}_\mathcal{F} \right) + \frac{B \cot \theta}{\sin^2 \theta} \mathcal{L}_\mathcal{F} &= 0 \end{aligned} \quad (\text{A5})$$

with

$$\begin{aligned} \mathcal{L}_\mathcal{F} &= \frac{d}{d\mathcal{F}} \mathcal{L}(\mathcal{F}), \\ \mathcal{F} &= -2E^2 + \frac{2B^2}{r^4 \sin^2 \theta}. \end{aligned} \quad (\text{A6})$$



Note that the lapse function  $f(r)$  disappears in the electromagnetic equations. We also note by passing that equations (A4), (A5) are unchanged if one replaces  $f(r)$  with  $f(r) - 2M/r$  where  $M$  is a constant. For regular black holes, this constant is always set to zero to avoid a curvature singularity at the origin [23, 24].

Now we are ready to go to spacetimes asymptotic to dS/AdS at spatial infinity  $r \rightarrow \infty$ . For concreteness, we will focus on the AdS case with a cosmological constant  $\Lambda = -3/\ell^2$ . Of course, it is easy to adjust our results to the dS case by reversing the signature of  $\ell^2$ .

Switching on the cosmological constant in action (A1) is equivalent to replacing  $\mathcal{L}(\mathcal{F})$  with  $\mathcal{L}(\mathcal{F}) - 6\ell^{-2}$ . What is more, one can check that equations (A4), (A5) are unchanged under the following replacement

$$\mathcal{L} \rightarrow \mathcal{L} - \frac{6}{\ell^2}, \quad f \rightarrow f + \frac{r^2}{\ell^2}, \quad E \rightarrow E, \quad B \rightarrow B. \quad (\text{A7})$$

That is to say, *in the Einstein gravity, from an asymptotically flat solution of the spherical static form (1), one can get an asymptotically AdS solution by replacing the lapse function  $f(r)$  with  $f(r) + r^2/\ell^2$ , leaving intact the electromagnetic field.*

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